Notes on Simulating Demand from Multiple Discrete/Continuous Models

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This note outlines a method for simulating demand from multiple discrete/continuous demand models. In this class of models, demand equations are often complicated expressions without a closed form, which complicates the process of simulating demand. We focus on a simulation approach based on analytical expressions of the Kuhn-Tucker conditions.

1 Preliminaries

There are two approaches for deriving models of consumer choice. The first is the indirect utility or "dual approach" which specifies an indirect utility function $V(p, M)$ and then employs Roy's Identity to derive demand functions. The second is the direct utility or "primal approach" which specifies a direct utility function $U(x)$ and budget constraint, and then employs the Kuhn-Tucker (KT) conditions of optimality to derive demand functions. This note will consider the latter direct utility approach and will specifically focus on multiple discrete/continuous (MDC) models à la Kim et al. (2002, 2007) and Bhat (2005, 2008).

MDC models get their name from the type of consumption patterns they admit: (i) consumption of multiple offerings; and (ii) consumption of continuous quantities. For instance, in a brand choice model based on discrete choice, consumption would be limited to, say, Lays potato chips. In a discrete/continuous model, consumers still choose one offering (Lays), but can consume continuous quantities – e.g., two bags of Lays chips. In an MDC model, consumption is not limited to the choice of a single offering, but can take the form of multiple offerings with each being consumed in continuous quantities – e.g., two bags of Lays potato chips and one bag of Ruffles potato chips.

The foundation of MDC models is the maximization of a direct utility function subject to a budget constraint:

$$\max_{x,z} U(x, z) \quad (1)$$
$$\text{s.t.} \quad p'x + z \leq E$$

where $x = (x_1, \ldots, x_J)$ is a consumption vector for the inside goods and z is the consumption of the (essential) outside good. We can solve for optimal quantities through the KT first-order conditions.

$$\begin{aligned} \frac{\partial U(x,z)}{\partial x_j} - \lambda p_j = 0 \quad \text{if } x_j^* > 0 \\ \frac{\partial U(x,z)}{\partial x_j} - \lambda p_j < 0 \quad \text{if } x_j^* = 0 \\ \frac{\partial U(x,z)}{\partial z} - \lambda = 0 \end{aligned} \quad (2)$$

When utility is specified as a nonlinear function of quantities x, then the resulting demand equations $x^*(p, E)$ will admit both corner and interior solutions.

There are many popular choices of utility functions, including translated CES or "power" utility (Kim et al., 2002) and translated Cobb-Douglas (Pollak and Wales, 1969). There are also "generalized" versions of each, which allow for more flexible expressions of marginal utility. In this note, we will focus on the generalized translated Cobb-Douglas utility (Hanemann, 1978; Bhat, 2008):

$$U(x, z) = \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) + u_z(z) \quad (3)$$

where ψ_j is a baseline marginal utility parameter, which is the value of marginal utility with $x_j = 0$, and γ_j is a satiation parameter, which controls the curvature of utility.

2 Simulation Algorithm

One challenge with MDC models is that the expressions of demand arising from (1) can be complicated, and may not have a closed form. For example, in the case of (1) with an outside good with nonlinear utility, then $x_j(p, E)$ will be an implicit function. This, in turn, makes it challenging to do things like generate data for a simulation study or forecast demand. While it is always possible to generate demand through constrained optimization routines, like `constrOptim` in R, such procedures often require imposing ad-hoc decisions about how to treat small, but positive quantities. In practice, simulations may be sensitive to these choices of what is a corner solution vs. what is an interior solution.

The goal of this note is to discuss an approach to simulate from MDC models that makes explicit use of the KT conditions. To my knowledge, this algorithm first appeared in Pinjari and Bhat (2011).[1] The idea is as follows. If we knew which and how many goods were selected, then deriving quantities would follow immediately from the KT conditions in (2). Even though we don't know this a priori, we can use the KT conditions to figure this out. To start, assume that only the outside good is chosen and then check whether the KT conditions in (2) hold. If not, then it must be the case that at least one $x_k > 0$, with k being the good with the highest "bang for the buck": $\psi_k/p_k \geq \psi_j/p_j$ for all $j = 1, \ldots, J$. So we can then rank goods based on their value of ψ_j/p_j and assume that z and only the first-ranked good is chosen. We then again check the KT conditions and if they are satisfied, stop; if not, then assume that z and the top two goods are chosen, etc. Formally, the algorithm can be written as follows.

1. Arrange J goods in order of ψ_j/p_j and define $M = 1$

2. Compute λ^*

3. If $\lambda^* > \psi_{M+1}/p_{M+1}$ then compute x_j^* for $j \leq M$ set $x_j^* = 0$ for $j > M$.
   Otherwise, go to next step.

4. If $M < J$, set $M = M + 1$ and return to step 2.
   Otherwise, compute x_j^* and stop.

3 Solving for λ^* and $x(\lambda^*)$

The algorithm above requires solving for the optimal Lagrange multiplier λ^* and demand as a function of this optimal value $x_j(\lambda^*)$. In this section, we solve for each of these objects under three scenarios: (1) a model without an outside good; (2) a model with an outside good with linear outside good utility; and (3) a model with

[1] A similar version was also proposed by Kevin Van Horn (then at The Modellers) in a 2012 technical report entitled "Algorithm for Volumetric Forecasting", which accompanies the discussion of MDC models in chapter 5 of the "Seven Summits of Marketing Research" textbook. This version has since been published in the appendix of Allenby et al. (2019).

an outside good with nonlinear outside good utility. For each type of model, we first write down the Lagrangian. We then write the KT conditions, write demand as a function of λ, plug $x(\lambda)$ into the budget constraint to solve for λ^*, and then plug λ^* back into the demand function to get $x(\lambda^*)$.

3.1 No Outside Good Model

$$\begin{aligned} &\max_x \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) \\ &\text{s.t.} \quad p'x \le E \end{aligned} \tag{4}$$

(i) Lagrangian

$$\mathcal{L} = \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) + \lambda(E - p'x) \tag{5}$$

(ii) Write Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \qquad \text{if } x_j^* > 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j < 0 \qquad \text{if } x_j^* = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - x'p = 0 \tag{8}$$

(iii) Write demand as a function of lambda

$$\frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \implies x_j(\lambda) = \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} \tag{9}$$

(iv) Plug into budget constraint and solve for optimal lambda

$$\begin{aligned} \sum_{j=1}^{J} p_j x_j = E &\implies \sum_{j=1}^{J} p_j \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} = E \\ &\implies \frac{1}{\lambda} = \frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j}} \\ &\implies \lambda^* = \left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j}} \right)^{-1} \end{aligned} \tag{10}$$

(v) Plug optimal lambda back into demand functions

$$\begin{aligned} x_k^* &= x_k(\lambda^*) \\ &= \frac{\psi_k - \lambda^* p_k}{\gamma_k \lambda^* p_k} \\ &= \frac{1}{\gamma_k}\left[\left(\frac{1}{\lambda^*}\right)\frac{\psi_k}{p_k} - 1\right] \\ &= \frac{1}{\gamma_k}\left[\left(\frac{E + \sum_{j=1}^{J}\frac{p_j}{\gamma_j}}{\sum_{j=1}^{J}\frac{\psi_j}{\gamma_j}}\right)\frac{\psi_k}{p_k} - 1\right] \end{aligned} \tag{11}$$

3.2 Linear Outside Good Model

$$\begin{aligned} &\max_{x,z} \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j}\log(\gamma_j x_j + 1) + \psi_z z \\ &\text{s.t.} \quad p'x + z \le E \end{aligned} \tag{12}$$

(i) Lagrangian

$$\mathcal{L} = \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j}\log(\gamma_j x_j + 1) + \psi_z z + \lambda(E - p'x - z) \tag{13}$$

(ii) Write Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \qquad \text{if } x_j^* > 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j < 0 \qquad \text{if } x_j^* = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \psi_z - \lambda = 0 \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - x'p - z = 0 \tag{17}$$

(iii) Write demand as a function of lambda

$$\frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \implies x_j(\lambda) = \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} \tag{18}$$

(iv) Plug into budget constraint and solve for optimal lambda

$$\begin{aligned}\sum_{j=1}^{J} p_j x_j + z = E &\implies \sum_{j=1}^{J} p_j \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} + z = E \\ &\implies \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j \lambda} - \sum_{j=1}^{J} \frac{p_j}{\gamma_j} + z = E \\ &\implies \frac{1}{\lambda} \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} = E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j} - z \\ &\implies \frac{1}{\lambda} = \frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j} - z}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \\ &\implies \lambda^* = \left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j} - z}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \right)^{-1}\end{aligned} \tag{19}$$

(v) Plug lambda back into demand functions

$$z^* = \psi_z \left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \right) \tag{20}$$

$$\begin{aligned}x_k^* &= x_k(\lambda^*) \\ &= \frac{\psi_k - \lambda^* p_k}{\gamma_k \lambda^* p_k} \\ &= \frac{1}{\gamma_k} \left[\left(\frac{1}{\lambda^*} \right) \frac{\psi_k}{p_k} - 1 \right] \\ &= \frac{1}{\gamma_k} \left[\left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j} - z}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j}} \right) \frac{\psi_k}{p_k} - 1 \right]\end{aligned} \tag{21}$$

3.3 Nonlinear Outside Good Model

$$\begin{aligned}&\max_{x,z} \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) + \psi_z \log(z) \\ &\text{s.t.} \quad p'x + z \leq E\end{aligned} \tag{22}$$

(i) Lagrangian

$$\mathcal{L} = \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) + \psi_z \log(z) + \lambda(E - p'x - z) \tag{23}$$

(ii) Write Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \qquad \text{if } x_j^* > 0 \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j < 0 \qquad \text{if } x_j^* = 0 \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{\psi_z}{z} - \lambda = 0 \tag{26}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = E - x'p - z = 0 \tag{27}$$

(iii) Write demand as a function of lambda

$$\frac{\psi_j}{\gamma_j x_j + 1} - \lambda p_j = 0 \implies x_j(\lambda) = \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} \tag{28}$$

(iv) Plug into budget constraint and solve for optimal lambda

$$\begin{aligned} \sum_{j=1}^{J} p_j x_j + z = E &\implies \sum_{j=1}^{J} p_j \frac{\psi_j - \lambda p_j}{\gamma_j \lambda p_j} + z = E \\ &\implies \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j \lambda} - \sum_{j=1}^{J} \frac{p_j}{\gamma_j} + \frac{\psi_z}{\lambda} = E \\ &\implies \frac{1}{\lambda} \left(\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z \right) = E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j} \\ &\implies \frac{1}{\lambda} = \frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \\ &\implies \lambda^* = \left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \right)^{-1} \end{aligned} \tag{29}$$

(v) Plug optimal lambda back into demand functions

$$z^* = \psi_z \left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \right) \tag{30}$$

$$\begin{aligned} x_k^* &= x_k(\lambda^*) \\ &= \frac{\psi_k - \lambda^* p_k}{\gamma_k \lambda^* p_k} \\ &= \frac{1}{\gamma_k} \left[\left(\frac{1}{\lambda^*} \right) \frac{\psi_k}{p_k} - 1 \right] \\ &= \frac{1}{\gamma_k} \left[\left(\frac{E + \sum_{j=1}^{J} \frac{p_j}{\gamma_j}}{\sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} + \psi_z} \right) \frac{\psi_k}{p_k} - 1 \right] \end{aligned} \tag{31}$$

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