Online Appendix for "Demand Models with Random Partitions"

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A Proofs

Property 1 (Location-Scale Consistency). *If $\pi_n \sim LSP(\rho_n, \tau)$, then for any number of items n and location partition $\rho_n \in \mathcal{P}_n$,*

$$\lim_{\tau \to 0} \Pr(\pi_n = \rho_n | \rho_n, \tau) = 1.$$

Proof. Since each item-group assignment variable g_i is generated sequentially, we use mathematical induction to show that $g_i = s_i$ as $\tau \to 0$ for $i = 1, \ldots, n$. Since $g_1 = s_1 = 1$ trivially, we use $i = 2$ as a base case.

BASE CASE: There are two cases to consider.

(i) Suppose $s_2 = s_1$. We must show that as τ approaches zero, the probability that item 2 starts a new group goes to zero ($w_0(\cdot) \to 0$) and the probability that it joins the first group goes to one ($w_1(\cdot) \to 1$).

$$w_0(s_2, \tau) = \tilde{c}_2 \cdot \frac{\tau + 1(s_2 = C^{(2)} + 1)}{\tau C^{(2)} + \tau + 1} = \tilde{c}_2 \cdot \frac{\tau}{\tau C^{(2)} + \tau + 1} \to 0$$
$$w_1(\{s_2, S_1\}, \tau) = \tilde{c}_2 \cdot \frac{\tau + n_{S_1}^{s_2}}{\tau C^{(2)} + \tau + n_1} = \tilde{c}_2 \cdot \frac{\tau + 1}{\tau C^{(2)} + \tau + 1} \to 1$$

(ii) Suppose $s_2 = C^{(2)} + 1 \neq s_1$. We must show that as τ approaches zero, the probability that item 2 starts a new group goes to one ($w_0(\cdot) \to 1$) and the probability that it joins the first group goes to zero ($w_1(\cdot) \to 0$).

$$
\begin{aligned}
w_0(s_2,\tau) &= \tilde{c}_2 \cdot \frac{\tau + 1(s_2 = C^{(2)}+1)}{\tau C^{(2)} + \tau + 1} = \tilde{c}_2 \cdot \frac{\tau+1}{\tau C^{(2)} + \tau + 1} \to 1 \\
w_1(\{s_2, S_1\},\tau) &= \tilde{c}_2 \cdot \frac{\tau + n_{S_1}^{s_2}}{\tau C^{(2)} + \tau + n_1} = \tilde{c}_2 \cdot \frac{\tau}{\tau C^{(2)} + \tau + 1} \to 0
\end{aligned}
$$

INDUCTIVE STEP: Assume that $g_i = s_i$ for $i = 1, \ldots, j-1$ where $j < n+1$. We wish to show that $g_j = s_j$. There are again two cases to consider.

(i) Suppose $s_j = c$ where $c \in \{1, \ldots, C^{(j)}\}$. We must show that as τ approaches zero, the probability that item j starts a new group goes to zero ($w_0(\cdot) \to 0$), the probability that it joins group c goes to one ($w_c(\cdot) \to 1$), and the probability it joins any other group $k \neq c$ goes to zero ($w_k(\cdot) \to 0$).

$$
\begin{aligned}
w_0(s_j,\tau) &= \tilde{c}_j \cdot \frac{\tau + 1(s_j = C^{(j)}+1)}{\tau C^{(j)} + \tau + 1} = \tilde{c}_j \cdot \frac{\tau}{\tau C^{(j)} + \tau + 1} \to 0 \\
w_c(\{s_j, S_c\},\tau) &= \tilde{c}_j \cdot \frac{\tau + n_{S_c}^{s_j}}{\tau C^{(j)} + \tau + n_c} = \tilde{c}_j \cdot \frac{\tau + n_c}{\tau C^{(j)} + \tau + n_c} \to 1 \\
w_k(\{s_j, S_k\},\tau) &= \tilde{c}_j \cdot \frac{\tau + n_{S_k}^{s_j}}{\tau C^{(j)} + \tau + n_k} = \tilde{c}_j \cdot \frac{\tau}{\tau C^{(j)} + \tau + n_k} \to 0
\end{aligned}
$$

(ii) Suppose $s_j = C^{(j)} + 1$. We must show that as τ approaches zero, the probability that item j starts a new group goes to one ($w_0(\cdot) \to 1$) and the probability that it joins group k goes to zero ($w_k(\cdot) \to 0$) for any $k = 1, \ldots, K^{(j)}$.

$$
\begin{aligned}
w_0(s_j,\tau) &= \tilde{c}_j \cdot \frac{\tau + 1(s_j = C^{(j)}+1)}{\tau C^{(j)} + \tau + 1} = \tilde{c}_j \cdot \frac{\tau+1}{\tau C^{(j)} + \tau + 1} \to 1 \\
w_k(\{s_j, S_k\},\tau) &= \tilde{c}_j \cdot \frac{\tau + n_{S_k}^{s_j}}{\tau C^{(j)} + \tau + n_k} = \tilde{c}_j \cdot \frac{\tau}{\tau C^{(j)} + \tau + n_k} \to 0
\end{aligned}
$$

□

Property 2 (Marginal Invariance). *If $\pi_n \sim LSP(\rho_n, \tau)$, then for any number of items n, location partition $\rho_n \in \mathcal{P}_n$, scale parameter $\tau > 0$, and distribution $p(s_{n+1})$ such that*

$\sum_{s_{n+1}} p(s_{n+1}) = 1$,

$$p(\pi_n|\rho_n, \tau) = \sum_{g_{n+1}=1}^{K+1} \sum_{s_{n+1}=1}^{C+1} p(\pi_{n+1}|\rho_n, s_{n+1}, \tau) p(s_{n+1})$$

where $K = \max\{g_1, \ldots, g_n\}$ *and* $C = \max\{s_1, \ldots, s_n\}$.

Proof. First pick an arbitrary value of $s_{n+1} \in \{1, \ldots, C+1\}$. By the sequential nature of the Pólya-urn scheme, we have

$$\begin{aligned}
\sum_{g_{n+1}=1}^{K+1} p(\pi_n, g_{n+1}|\rho_n, s_{n+1}, \tau) &= \sum_{g_{n+1}=1}^{K+1} p(g_{n+1}|\pi_n, \rho_n, s_{n+1}, \tau) p(\pi_n|\rho_n, s_{n+1}, \tau) \\
&= p(\pi_n|\rho_n, \tau) \sum_{g_{n+1}=1}^{K+1} p(g_{n+1}|\pi_n, \rho_n, s_{n+1}, \tau) \\
&= p(\pi_n|\rho_n, \tau) \big[w_0(\cdot) + w_1(\cdot) + \ldots + w_K(\cdot) \big] \\
&= p(\pi_n|\rho_n, \tau).
\end{aligned}$$

Since $p(\pi_n|\rho_n, \tau)$ does not depend on s_{n+1} and $\sum_{s_{n+1}=1}^{C+1} p(s_{n+1}) = 1$, it follows that

$$\begin{aligned}
\sum_{g_{n+1}=1}^{K+1} \sum_{s_{n+1}=1}^{C+1} p(\pi_n, g_{n+1}|\rho_n, s_{n+1}, \tau) p(s_{n+1}) &= \sum_{s_{n+1}=1}^{C+1} \left[\sum_{g_{n+1}=1}^{K+1} p(\pi_n, g_{n+1}|\rho_n, s_{n+1}, \tau) \right] p(s_{n+1}) \\
&= \sum_{s_{n+1}=1}^{C+1} p(\pi_n|\rho_n, \tau) p(s_{n+1}) \\
&= p(\pi_n|\rho_n, \tau) \sum_{s_{n+1}=1}^{C+1} p(s_{n+1}) \\
&= p(\pi_n|\rho_n, \tau)
\end{aligned}$$

as desired. □

B Behavior of the ddCRP and EPA Models

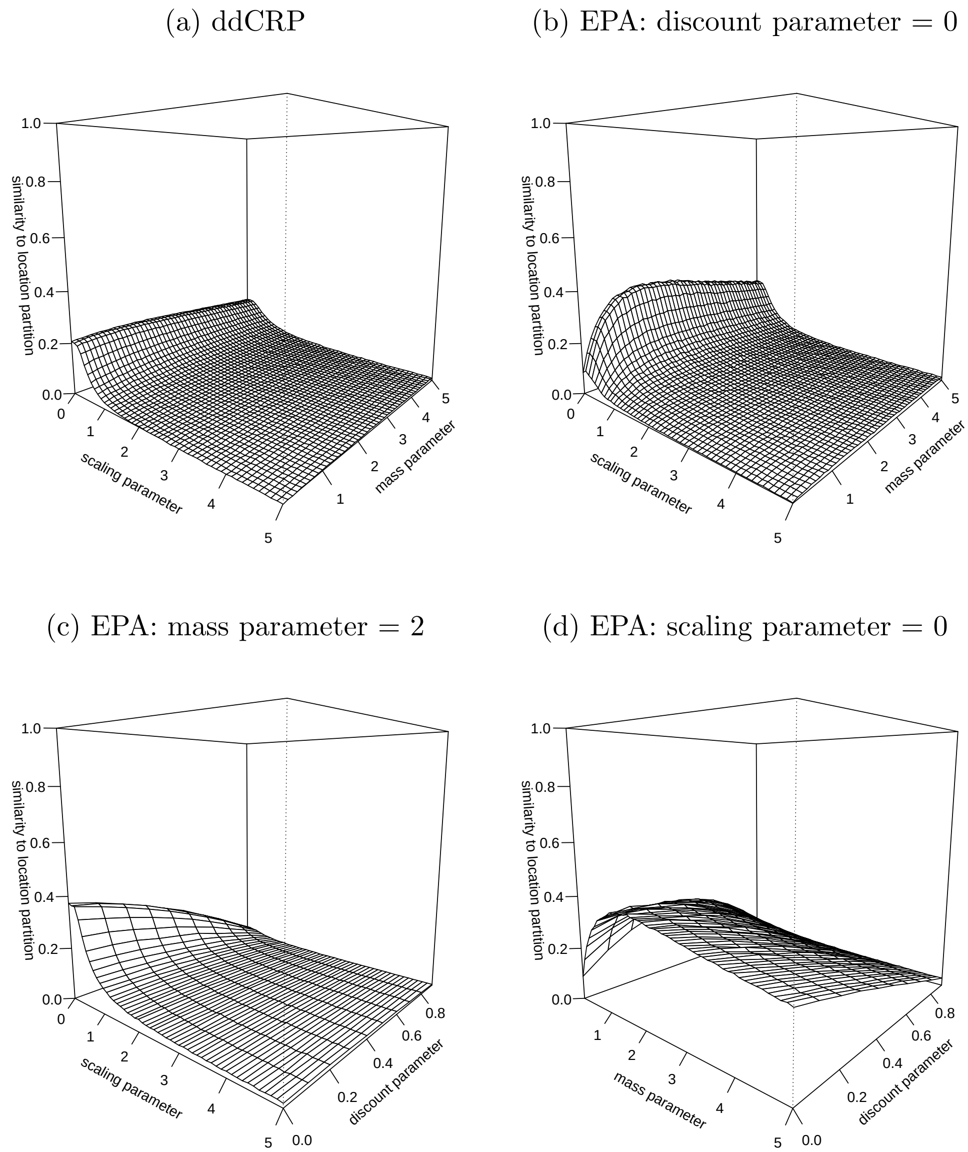

Figure B.1: Each plot shows the extent to which the ddCRP and EPA distributions can be centered around a location partition ρ_n. For each partition distribution, 10,000 random partitions of length $n = 10$ are drawn and then compared to $\rho_n = (1, 1, 1, 1, 1, 2, 2, 2, 2, 2)$ using the adjusted Rand index. The EPA and ddCRP distributions are both parameterized by an exponential decay function and by the pairwise distance matrix induced by ρ_n. The surface of each plot then shows the averaged adjusted Rand index across values of the scaling, mass, or discount parameters.

C Product Descriptions

Table C.1: Salty Snack Product Descriptions

	Brand	Subcategory	Subcategory Volume Share	Feature Frequency	Display Frequency
1	BARREL O FUN	CHESNK	23.24	1.56	31.97
2	CHEETOS	CHESNK	76.76	13.13	29.43
3	BARREL O FUN	CRNSNK	13.12	0.48	66.46
4	BUGLES	CRNSNK	19.17	18.63	20.53
5	FRITOS	CRNSNK	38.42	22.14	30.84
6	FRITOS SCOOPS	CRNSNK	26.97	0.18	34.04
7	OLD DUTCH	CRNSNK	2.32	0.00	1.91
8	GARDETTOS	OTHER	12.06	19.41	11.24
9	GENERAL MILLS CHEX MIX	OTHER	47.07	16.84	18.49
10	MUNCHOS	OTHER	1.26	0.00	0.00
11	PRIVATE LABEL	OTHER	2.15	0.00	0.00
12	S & W PIK NIK	OTHER	3.48	0.00	0.00
13	SUNCHIPS	OTHER	33.99	21.05	47.58
14	BAKED LAYS	PTOCHP	2.48	0.48	19.25
15	BAKED RUFFLES	PTOCHP	2.45	6.70	19.40
16	BARREL O FUN	PTOCHP	4.69	5.96	27.02
17	LAYS	PTOCHP	27.14	35.09	58.54
18	OLD DUTCH	PTOCHP	9.07	4.20	12.03
19	POORE BROTHERS	PTOCHP	3.78	0.96	21.25
20	PRINGLES	PTOCHP	14.89	5.47	6.34
21	PRINGLES CHEEZUMS	PTOCHP	1.81	5.26	3.83
22	PRINGLES FAT FREE	PTOCHP	1.40	0.00	0.00
23	PRINGLES RIGHT CRISPS	PTOCHP	1.76	4.88	4.81
24	PRIVATE LABEL	PTOCHP	8.80	9.61	11.35
25	RUFFLES	PTOCHP	9.93	13.24	33.17
26	WAVY LAYS	PTOCHP	11.81	34.98	51.96
27	BARREL O FUN	PRETZL	4.07	0.00	25.84
28	OLD DUTCH	PRETZL	12.42	11.31	0.69
29	PRIVATE LABEL	PRETZL	27.19	13.46	14.99
30	ROLD GOLD	PRETZL	45.32	11.09	37.28
31	SNYDERS OF HANOVER	PRETZL	11.00	3.40	16.78
32	BARREL O FUN	POPCRN	28.08	0.00	2.75
33	CRUNCH N MUNCH	POPCRN	25.56	20.93	0.00
34	OLD DUTCH	POPCRN	46.37	1.68	0.00
35	BAKED TOSTITOS	TTACHP	3.03	0.96	20.57
36	BARREL O FUN	TTACHP	18.21	2.92	32.96
37	DORITOS	TTACHP	40.57	25.00	58.92
38	GARDEN OF EATIN BLUE CHIPS	TTACHP	1.14	3.53	7.02
39	OLD DUTCH	TTACHP	5.07	7.49	14.74
40	TOSTITOS	TTACHP	31.98	12.90	37.84

D MH Step for the Isolated Demand Model

1. Generate the candidate partition

$$\pi_n^* \sim q_1(\pi_n | \pi_n^{(r)}, v) = \mathrm{LSP}(\pi_n^{(r)}, v)$$

where $v = 1/(n \log(n))$. Then conditional on π_n^*, generate $\boldsymbol{\beta}_{\pi_n}^*$ from its full conditional distribution

$$\boldsymbol{\beta}_{\pi_n}^* \sim q_2(\boldsymbol{\beta} | \boldsymbol{y}, \boldsymbol{X}_{\pi_n^*}, \pi_n^*, \Sigma) = \mathrm{N}(\tilde{\boldsymbol{\beta}}, (\tilde{\boldsymbol{X}}'_{\pi_n^*} \tilde{\boldsymbol{X}}_{\pi_n^*} + A_{\pi_n^*})^{-1})$$

where $\tilde{\boldsymbol{\beta}} = (\tilde{\boldsymbol{X}}'_{\pi_n^*} \tilde{\boldsymbol{X}}_{\pi_n^*} + A_{\pi_n^*})^{-1} (\tilde{\boldsymbol{X}}'_{\pi_n^*} \tilde{\boldsymbol{y}} + A_{\pi_n^*} \bar{\boldsymbol{\beta}}_{\pi_n^*})$, $\tilde{\boldsymbol{X}}_{\pi_n^*} = ((U^{-1})' \otimes I) \boldsymbol{X}_{\pi_n^*}$, and $\Sigma = U'U$.

2. Set $(\pi_n^{(r+1)}, \boldsymbol{\beta}_{\pi_n}^{(r+1)}) = (\pi_n^*, \boldsymbol{\beta}_{\pi_n}^*)$ with probability

$$\begin{aligned}
&\mathcal{A}(\pi_n^*, \boldsymbol{\beta}_{\pi_n}^*, \pi_n^{(r)}, \boldsymbol{\beta}_{\pi_n}^{(r)}) \\
&= \min \left\{ 1, \frac{p(\boldsymbol{y} | \boldsymbol{X}_{\pi_n^*}, \boldsymbol{\beta}_{\pi_n}^*, \pi_n^*, \Sigma) p(\boldsymbol{\beta}_{\pi_n}^* | \pi_n^*) p(\pi_n^*)}{p(\boldsymbol{y} | \boldsymbol{X}_{\pi_n^{(r)}}, \boldsymbol{\beta}_{\pi_n}^{(r)}, \pi_n^{(r)}, \Sigma) p(\boldsymbol{\beta}_{\pi_n}^{(r)} | \pi_n^{(r)}) p(\pi_n^{(r)})} \times \frac{q_2(\boldsymbol{\beta}_{\pi_n}^{(r)} | \pi_n^{(r)}) q_1(\pi_n^{(r)} | \pi_n^*)}{q_2(\boldsymbol{\beta}_{\pi_n}^* | \pi_n^*) q_1(\pi_n^* | \pi_n^{(r)})} \right\}.
\end{aligned}$$

Otherwise, set $(\pi_n^{(r+1)}, \boldsymbol{\beta}_{\pi_n}^{(r+1)}) = (\pi_n^{(r)}, \boldsymbol{\beta}_{\pi_n}^{(r)})$.